Mathematics from Fluids

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Abstract: *'Mathematics from Fluids' is a work in progress which currently takes the form of a series of posters which aim to develop an understanding of the physical implications of the Navier-Stokes equations based on an understanding of basic Fluid Mechanics principles. Taking the principle of Archimedes as a starting point, the mathematical relationships which build into the N-S equations, are developed using only the laws of Hydrostatics, Continuity and Conservation of Momentum. Thus the mathematics is derived from the laws of fluid mechanics rather than the other way round. The primary aim of the poster series is to help engineering students to appreciate both the concepts embodied in the N-S equations and the symbiotic relationship which exists between the realms of fluid mechanics and mathematics. Each poster is backed up by a short Microsoft Power point presentation which illustrates the development of the ideas presented.*

Introduction

Deriving mathematical concepts from a consideration of fluid behaviour clearly turns the conventional concept of derivation on its head. Thus the approach calls for some justification. It is this that provides the basic thrust for this paper.

The conventional route involves deriving a general result in mathematics by the logical extension of previously established mathematical constructs and techniques. In this way mathematic ideas are extended to ever higher levels of abstraction thereby providing the very powerful tools that we apply the many different fields of engineering. This approach has clear advantages in leveraging the skills and knowledge base acquired as it allows the techniques, once learned, to be applied in widely differing circumstances. The disadvantage is that, at least for the less gifted student, the details of a particular engineering problem may appear to bear little relation to the context in which the mathematics used to solve it was originally presented. As an example one might cite a typical course on the calculus where great emphasis is placed on solving "tricky" problems of integration or differentiation the reason for which becomes manifest to the student only when a particular example is re-visited in an engineering context. Further, a relationship which might be quite apparent as it arises in the context of a particular problem may be quite obscure to the student when presented as arising from a series of abstract mathematical concepts. As S.S. Sazihin mentions in his paper "The analysis of self-assment forms completed by students show that their progress in understanding physical concepts is much more visible than than their progress in understanding mathematical concepts"(Sazhin, 1998).

My first example of obtaining a mathematical result from a fluid mechanics principle is based on the principle of Archimedes, chosen because it is both widely understood and generally accepted. This principle may be stated in words as, "When a solid is immersed in a liquid it experiences an upthrust force which is equal to the weight of the liquid displaced".

The mathematical result obtained is the theorem of Gauss as it applies to the hydrostatic pressure in a fluid. In this way the simple act of proving Archimedes, result is leveraged to obtain a more fundamental mathematical result in an engineering context.

My second example which follows from the first extends the idea to from the scalar pressure field of hydrostatic to the vector field of flow velocity where the technique is used to apply the continuity principle of fluid mechanics to an arbitrarily shaped control volume.

The final example in this set extends the previous result to the vector field of momentum where it is used to apply the law of conservation of momentum to the control volume.

The material

The original aim of this work was to incorporate the use of vector notation in the teaching of basic fluid mechanics including Archimedes' principle, the continuity principle and the momentum principle. The perceived benefits of this approach are:

1. To reduce the constraints which normally apply to the selection of a suitable control volume.
2. To demonstrate the power of vector analysis as a means of developing fundamental fluid mechanics concepts.

The work is presented as a series of four posters (Swann 2008) with each section of a poster hyperlinked to a short explanatory Microsoft Power Point presentation. An example is shown below.

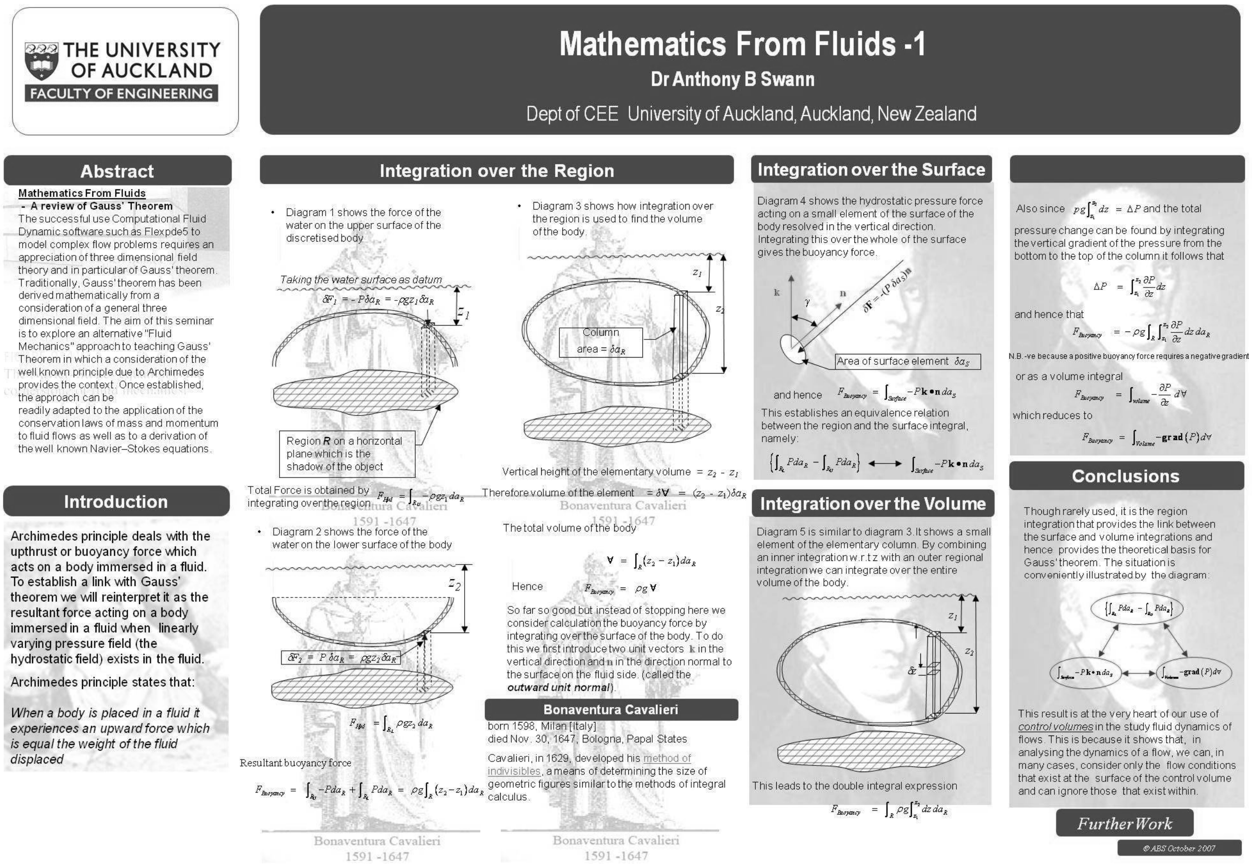

Figure 1: Poster 1 - Three approaches to the proof of Archimedes' principle

The first poster compares three approaches to the proof of Archimedes' principle using three distinct integration techniques. The first technique is integration over a region created by projecting the body onto a horizontal surface. Here the body is visualised as a collection of columns with each one being defined by an element of the region. This is the classical approach to the proof of Archimedes principle. The second approach is integration of the pressure force over the surface of the body. This makes use of the scalar or dot product applied to an element of the surface of the body. The third approach is to combine the integration over the region with an integration of the pressure gradient over the height of the column. Hence this integration is effectively carried out over the entire volume of the body. At this point we may conclude that, since all three integrations should yield the same result they must necessarily be equivalent to one another. Figure 1 shows the result as an equivalence triangle.

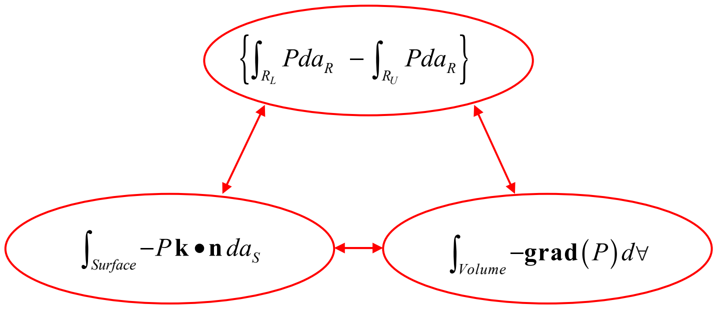

Figure 2: Equivalence triangle for Archimedes

In the second poster and its associated Microsoft Power Point slides, the equivalence relations established in Poster 1 are used to establish the continuity principle of fluid mechanics. For this purpose the submerged body is replaced by a control volume also of general shape while the hydrostatic pressure field is replaced by the general velocity vector field. However, since the equivalence relations apply to a scalar field it is necessary to apply them separately to the components of the velocity vector. Hence for this application the pressure scalar is replaced in turn by each velocity component. Then multiplication of the elementary area by the velocity component perpendicular to the elementary area gives the rate of flow crossing the element. Then integration using either region surface or volume integration results in the net rate of flow out of the control volume. We may note that the equivalence triangle for continuity includes Gauss' theorem (See Figure 2).

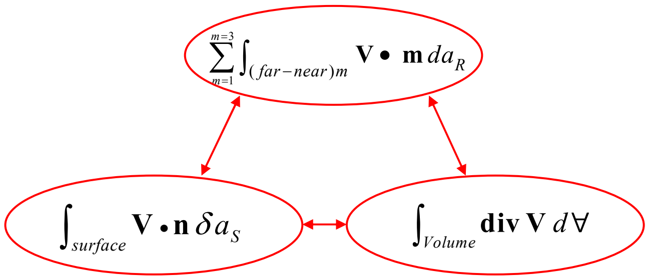

Figure 3: Equivalence triangle for continuity

In the third poster the momentum principle is applied to a control volume of general shape. In considering the conservation of momentum it is necessary to consider both the change in momentum flux and the forces responsible for the change. The former is relatively straightforward as we may simply take the rate of flow through each element and multiply by the density of the fluid to obtain the required momentum flux so for this part the equivalence relations are very similar to the relations used in considering continuity. However handling the external forces acting requires other techniques. The three external forces which are considered in a basic consideration of the subject are those due to gravity, pressure and viscosity of the fluid. Since gravity on all of the matter within the control colume the net gravity force can only be obtained by integrating over the volume. Pressure and viscous forces on the other hand are considered to be acting on the surface of the control volume. The consideration of the pressure force is identical in most respects to that used in the proof of Archimedes principle except that pressure variations in all three coordinate directions must be considered. The consideration of the viscous force is based on Newton's concept which he termed "lack of slipperiness" and described thus: "The resistance which arises from the lack of slipperiness originating in a fluid which, other things being equal, is proportional to the velocity by which the parts of the fluid are being separated from each other" (Dooge, 1983). In other words, viscous stress is proportional to the rate of strain of the fluid.

If we again consider the surface to be made up of a large number of steps and risers then the rate of separation of parts of the fluid from each other can be written in terms of the local velocity gradients. For instance, in a direction perpendicular to a horizontal step, the rate of separation is simply δVz/δz or in other words grad(V_z)•k and in the other two directions will be grad(V_x)•i and grad(V_y)•j. The consequent viscous force on the step is obtained by multiplying by the tread area δa_R and the coefficient of proportionality μ. Again the total force generated can be obtained by regional, surface or volume integration. Poster 3 shows the result obtained by applying volume integration to the z-direction component of the total viscous force.

$$\frac{F_{z(Viscosity)}}{\mu} = \int_{Volume} \left\{ \begin{array}{c} \dfrac{\partial\left(\mathbf{grad}\left(V_z\right)\right)_x}{\partial x} \\ +\dfrac{\partial\left(\mathbf{grad}\left(V_z\right)\right)_y}{\partial y} \\ +\dfrac{\partial\left(\mathbf{grad}\left(V_z\right)\right)_z}{\partial z} \end{array} \right\} d\forall$$

Or using the operator del

$$F_{z(Viscosity)} = \mu\int_{Volume} \nabla^2\left(V_x\right) d\forall$$

The conclusion to Poster 3 equates the change in momentum to the vector sum of the external forces

$$\begin{aligned} \int_{vol} \mathbf{V}\bullet\mathbf{grad}\left(\rho V_z\right) d\forall = & \int_{vol} \rho \boldsymbol{Z} d\forall \\ & + \int_{vol} -\mathbf{grad}\left(P\right)\bullet\mathbf{k} d\forall \\ & + \mu\int_{vol} \mathbf{div}\left(\mathbf{grad}\left(V_z\right)\right) d\forall \end{aligned}$$

Before leaving Poster 3 it may be worth noting that whereas the poster has generally made use of volume integration as this leads to a recognisable version of the Navier-Stokes equation there is no particular reason why the integration types could not be mixed if this happened to be more straightforward.

Poster 4 is in two parts. In the first part the results from Poster 3 are re-worked into the familiar form of the Navier-Stokes equations as presented in most modern fluid mechanics textbooks (Cengel & Cimbala, 2006). In the second part, the force relationships obtained in Poster 3 using the control volume approach are equated to the surface forces acting on an identical fluid particle, The reason for doing this is related to the mathematically based derivation of the Navier-Stokes equations which makes use of the so called "Constitutive equations". The constitutive equations relate the surface forces acting on a fluid particle to the state of stress present in the fluid. In Poster 4 we find that by simply equating the forces on the surfaces of the control volume with the equivalent forces acting on the fluid particle and integrating it is a relatively straightforward matter to derive all nine constitutive equations:

$$\sigma_x = \left(-P + 2\mu\frac{\partial V_x}{\partial x}\right) \qquad \sigma_y = \left(-P + 2\mu\frac{\partial V_y}{\partial y}\right) \qquad \sigma_z = \left(-P + 2\mu\frac{\partial V_z}{\partial z}\right)$$

$$\tau_{zx} = \mu\left(\frac{\partial V_x}{\partial z} + \frac{\partial V_z}{\partial x}\right) \qquad \tau_{xy} = \mu\left(\frac{\partial V_y}{\partial x} + \frac{\partial V_x}{\partial y}\right) \qquad \tau_{yz} = \mu\left(\frac{\partial V_z}{\partial y} + \frac{\partial V_y}{\partial z}\right)$$

$$\tau_{yx} = \mu\left(\frac{\partial V_x}{\partial y} + \frac{\partial V_y}{\partial x}\right) \qquad \tau_{zy} = \mu\left(\frac{\partial V_y}{\partial z} + \frac{\partial V_z}{\partial y}\right) \qquad \tau_{xz} = \mu\left(\frac{\partial V_z}{\partial x} + \frac{\partial V_x}{\partial z}\right)$$

Conclusion

The Navier-Stokes equations are derived from the volume integral form of the momentum principle equations applied to an elementary oblong control volume. This is in contrast with the classical approach to the application of the momentum principle which in effect uses the surface integration over a piecewise continuous control surface chosen to make the integration process as straightforward as possible.

The practical difference is that the Navier-Stokes solution will give information about velocities, pressures etc everywhere in the flow field, in other words, to "model" the flow. It is this difference which is exploited when using numerical integration methods such as the finite element methods. Conventional momentum principle solutions yield only bulk results which apply to the whole control volume.

The constitutive equations allow us to use the velocity gradient information from the Navier-Stokes solution to find the distribution of normal and shear forces throughout the field.

Reflections

The mathematics from fluids materials has now been available for two academic years and so far has had a somewhat mixed reception by Fluids Mechanics 1 classes. While the students are clearly appreciative of the effort to integrate the study of mathematics and fluids mechanics they are usually concerned that success in fluid mechanics may become too dependent on their mathematical ability. On the plus side, familiarity with the divergence theorem at this stage is helpful for the later introduction of stream functions. They are then able to demonstrate a grasp of the vector calculus concepts within a fluid mechanics context. The divergence theorem in particular provides a valuable stepping-stone to the introduction of the concept of stream functions.

References

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