Introduction

At The University of Western Australia, similarly to many others, engineering students are expected to learn fundamental concepts in mathematics in their first year and to apply them later in engineering. Students often struggle with these concepts and cannot foresee their relevance to engineering. This study contributes understanding that will be important to educators seeking to address this problem. Whereas other studies, including a thorough approach at our university, have coordinated mathematics topics and their engineering applications as part of curriculum development, this study delves deeper than previous studies by investigating how the mathematical concepts are presented and explained in mathematics and engineering.

The study is framed by threshold concept theory. Within this framework it is understood that curriculum developers can improve student learning by identifying the concepts that are most transformative and critical to future learning or practice in the discipline, namely 'threshold concepts', and ensuring that students have sufficient opportunity and support to develop understanding of these (Meyer & Land, 2003). This study follows recommendations based on an Australian study to identify engineering foundation threshold concepts, in which many mathematical threshold concepts were identified (Male, 2012). Engineering students in that study identified the abstract nature of mathematics, and the apparent lack of relevance to engineering, as troublesome features of threshold concepts in mathematics. They suggested indicating the engineering application of mathematical concepts at the time each is first introduced to students (Male, 2011; Male & Baillie, 2014). The idea to integrate engineering applications when teaching mathematics has been adopted or recommended elsewhere in Australia, and in Europe and the USA (Güner, 2013; Hirst et al., 2004; Wandel, 2010).

The obvious way to integrate engineering applications when teaching mathematics is to include engineering applications in mathematics classes. However, informed by threshold concept theory, we asked whether the issue is complicated by additional factors beyond lack of examples demonstrating the relevance of the mathematics. Threshold concept theory proposes that by identifying the troublesome features of concepts, educators can develop initiatives to support students to overcome the thresholds. Threshold concepts can be troublesome for any reason and several common reasons are identified in threshold concept literature. In addition to other reasons, students can find concepts troublesome due to abstract knowledge, language, complexity, unfamiliar ways of thinking, and features that are counter-intuitive (Perkins, 2006). Troublesome language can include new language or language used differently from familiar usage (Meyer & Land, 2003, p. 9). We analysed the language of mathematics including the terminology and notation, comparing that used where concepts are taught in first year mathematics with that used when the concepts are applied in engineering.

We asked:

1. What are some real-world engineering examples that can be used in first year mathematics to demonstrate the relevance of the mathematics?

2. How is the language used to present mathematical concepts when they are applied in engineering similar to and different from the language with which the concepts are first introduced to engineering students in mathematics?

3. How do students respond to the identified mathematical concepts in engineering and mathematics as they are presented and explained?

Method

The first author of this paper is a mathematician and engineer in an engineering school and teaches second year, third year, and masters (fifth year) engineering units. She collaborated with the third author who was teaching mathematics to first year engineering students, in order to connect the teaching and application of mathematics between their units and thereby support students' learning. For certain topics in a first year mathematics unit, we identified relevant and important applications in a second, third year engineering and Masters of Professional Engineering unit. The third author peer-reviewed at least two lectures and one tutorial taught by the first author in both 2012 and 2014 in the third year unit Reservoir Characterisation, and also in 2014 in the second year unit Motion and masters unit Petroleum Engineering. The first author peer-reviewed three first year mathematics lectures and a two-hour tutorial of the third author. The foci of the peer reviews were how mathematics topics were presented and how students responded to this. The peer-reviewed classes were selected to include teaching or application of the same important mathematical concepts. Significantly, the first author's interdisciplinary background enabled a detailed analysis of the mathematical concepts in both disciplines. The first and third authors noted student reactions such as questions, comments and difficulties.

Findings

We found differences between the notation used in mathematics and engineering, and that the engineering units employed mathematical tools with little reference to the material learned in first year. Potential for improving the connections between mathematics and applications were identified for flow rate or flux (which applies to calculation of velocity), binomial distributions (which apply to calculation of effective permeability of a sample), and coordinate transformations (with application to calculation of directional permeability). The binomial distribution was applied in petroleum engineering in mixtures of random variables in geological sediments and notation from mathematics required further explanation before application in order for the students to recognise it. Coordinate transformations were applied and derived in petroleum engineering to rotate coordinates when permeability was measured in core sample plugs of various orientations.

The example of flux or flow rate is described below. Following engineering applications, we present the mathematics as it appeared in a first year mathematics unit, and describe how we addressed the inconsistencies.

Flux is the change of a quantity over a surface, often per unit time. Many types of flux are calculated in engineering. Applications in the units investigated in this study included mass flow rate and thermal flux.

In the second year engineering unit Motion mass flow rate was calculated and used in the conservation of mass or mass balance equation. The rate of accumulation of mass within a system equals the sum of the mass flow rates into the system minus the sum of the mass flow rates out of the system.

$$\frac{dm_{sys}}{dt} = \sum_{in} \dot{m}_i - \sum_{out} \dot{m}_e \tag{1}$$

where $\dot{m}$ is the mass flow rate or rate at which mass cross the boundary in kg/s.

In equation 1, students were presented with the dot notation that is commonly used in engineering to indicate differentiation with respect to time *t.* While this is likely to be tacit to engineering academics, several engineering students were not familiar with it. Flux was presented in this application as summation rather than integration.

Equation 1 presents the concept of a conservation and accounting principle in a rate form. Mass is a scalar variable and in equation 1 the system is represented without dimensions. The formal representation of this idea requires integration over the volume and boundary of the system. In the engineering unit this appeared later in the example of velocity calculation.

$$\frac{dm}{dt} = \int_A \rho(x,y,z) v_n \, dA \qquad (2)$$

where ρ is the density of the fluid (kg/m^3) and v_n is the normal component of the velocity vector (m/s).

In the third year and masters level petroleum engineering units, thermal flux was presented in Fourier's Law:

$$\underbrace{\frac{\Delta Q}{A\Delta t}}_{\text{Thermal flux}} = -k \underbrace{\frac{\Delta T}{\Delta x}}_{\text{Temperature gradient}} \qquad (3)$$

where:

k is thermal conductivity (Wm^{-1}K^{-1}).

A is the area of the surface through which heat flows, normal to the direction of flow which is aligned to x.

Q is heat (J).

T is temperature (K).

$\frac{\Delta Q}{\Delta t}$ is heat rate (W), with the negative sign indicating direction of flow.

In equation 3, flux was presented to students using increments rather than derivatives and no integral was necessary. The increments were used instead of derivatives, to facilitate visualisation of the concept.

Despite no integral appearing in either of the above two cases where flux appeared, "flux across a surface" was introduced to students in mathematics in first year as follows (Bassom et al., 2014, p. 73).

> Given a vector field **F** of R^3 and some surface S with a parametric representation **S**(u, v) for $(u, v) \in D$, the flux across S (in the standard direction of the unit normal vector **n**) is

$$\int_{S} \boldsymbol{F} \cdot \boldsymbol{dS} = \iint_{\boldsymbol{D}} \boldsymbol{F} \cdot \boldsymbol{N} du dv \qquad (4)$$

If C is a closed surface, we make sure that **N** points outwards (by interchanging the roles of *u* and *v* if needed).

The notation used in the first year mathematics unit (equation 4) was thus very general as it allowed for a vector field and a generic surface. It could be adapted readily to any specific application in engineering. Examples of adaptions are given in equations 1, 2 and 3. Other examples of applications in engineering include topics such as electrodynamics where magnetic flux is calculated.

Students indicated that they found the mathematical definition very abstract. Though some schematic visualisations were provided in the mathematics unit, the future relevance to specific three dimensional visualisations was not yet apparent to students. Moreover, connections of the mathematical definition to the applications in engineering were often very difficult for students to see. For example, the mathematics unit sometimes used a double integral to represent integration over the surface (right hand side of equation 4), whereas in the Motion unit a single integral was used (equation 2). Students described the single integral for integration over a surface as conflicting with their intuition that the single integral was used for integration along a curve and not a surface.

The mathematical concept of flux across a surface was developed visually in the mathematics unit, based on ideas of the Riemann integral. Here, increments in *u* and *v* directions were considered. Some students reported in the Motion unit that they considered these explanations to be solely relevant to understanding the mathematical foundations and therefore they tended to disregard the explanations and memorised the final formula. However, the importance of the mathematical explanation can be seen from the application of the increments in Fourier's Law (equation 3).

Discussion

Recommendations

This study revealed that the links between different notations and explanations used in mathematics and engineering were not clear for the students in the classes taught by the first and third authors. A possible source of the problem is that the links are tacit for academics and therefore not presented explicitly. This could lead to further unrecognised complications for students if academics in the two disciplines do not discuss their teaching with each other so that they are aware of similarities and differences between the notations and conceptual explanations that they use. We recommend continued communication between academics in mathematics and engineering regarding their teaching, especially notations and explanations used and the applications of mathematics in engineering. Interdisciplinary workshops involving academics who teach into engineering programs from various scientific disciplines and in various engineering fields could stimulate academics to coordinate their teaching not only at the topic-level but also such that students are supported in recognising connections within programs and diversity in terminology and notation is clarified for students.

If not consistently represented, concepts *and how they are presented and explained* could be cross-referenced between units. Eccles' expectancy value theory explains that one of the factors that motivates people towards a task is 'perceived utility value' which is perceived value for the person's future (Brown, McCord, Matusovich, & Kajfez, 2014). If academics in mathematics refer to future relevance, not just of the topic in general but the conceptual understanding of the explanation, this could motivate students to follow the explanations, and also equip them with the skill to adapt the concepts to future applications.

By teaching mathematics with engineering, rather than referencing engineering applications when teaching mathematics, it might be possible to reduce the problem identified in this paper. Hennig, Mertsching, and Hilkenmeier (2015) taught mathematics alongside engineering in a fundamental electrical engineering unit, to address issues of diversity in levels of mathematics studied among the engineering students. For similar reasons, Bhathal (2015) provided online tutorial systems to support students to revise relevant mathematical concepts before new engineering physics topics were introduced.

This investigation has drawn attention to a problem that requires rigorous investigation to be understood more thoroughly in order to design and test interventions. Better understanding of the thresholds that students face in applying mathematical concepts in engineering will require further investigation of how students experience the development of understanding of mathematical concepts at various stages in their education programs.

Baillie, Bowden, and Meyer (2013) combined threshold concept theory and capability theory to develop threshold capability theory. A student who has developed a threshold capability can respond to an unseen problem by identifying the significant features of the situation and developing and implementing a plan to build on relevant knowledge and respond successfully to the situation. Threshold capabilities rely on one or more threshold concepts. Our vision is for engineering students to have the capability to approach an unseen engineering problem and be able to recognise the salient features of the problem and identify and apply relevant mathematics including mathematical threshold concepts. Consistent with this perspective, Booth (2008) recommends that engineering students be supported in developing the ability to address a problem by working out what they can use of what they know and what they need to know. A repertoire of mathematical tools including a variety of notations is likely to be valuable for this purpose. The vision raises questions for further investigation.

While consistency in notation may simplify students' learning, there are also potential benefits to demonstrating diverse representations especially where these indicate nuanced conceptual understanding. Leppävirta (2011) recommends greater focus on conceptual understanding. It is not clear whether consistent notation or diverse notation would be most helpful in achieving this. Furthermore, Gainsburg (2015) draws attention to the importance of engineering students developing different levels of epistemological views towards mathematics methods, and recognising links between diverse approaches is part of this development.

Considering these perspectives on mathematics education for engineering, in responding to the preliminary findings of this study, discussion and cross-referencing between units could be preferable to uniform selection of notations and terminology. We expect it will be most valuable to encourage students to discuss, connect, and try to use various notations and consider the features of situations in which they are most suitable.

Limitations and Further Research

This investigation found troublesome features of applications of mathematical concepts experienced by engineering students in the units investigated. To develop curriculum improvements, further study is needed to investigate students' experiences of understanding and applying mathematical threshold concepts.

Further research will be required to evaluate the impact of efforts to improve links between mathematics and engineering. Exercises in which students identify and discuss connections between mathematics and various applications in engineering could be developed to both collect data and facilitate capability development by the students.

Further investigation of the nuances in conceptualisation reflected in the differences between the language notation used in mathematics and in engineering could also lead to deeper understanding and improved collaboration between the disciplines.

The applications of mathematics identified in this study were in mechanical and petroleum engineering units. Examples in other engineering disciplines should also be identified, especially as engineering is often taught with common foundation units in mathematics

Conclusions

In this study we have identified examples of engineering applications of mathematics and discovered that there are significant differences in notation and explanations in mathematics as taught in mathematics and applied in engineering. Representations in mathematics are abstract and general compared with those used in engineering. Identifying engineering examples for the mathematical concepts may not provide sufficient support for students. Possible strategies to support students in overcoming thresholds in mathematics are to improve the consistency in notation, or to introduce cross references between units including notations and explanations of the mathematical concepts. Discussion and trial of alternative representations and their suitability for various contexts could also be found to be valuable. We recommend that academics who teach mathematics to engineering students, and those who teach engineering units, discuss how they teach and apply mathematical concepts. They should delve beyond identifying mathematical concepts and applications in engineering, and consider development of levels of abstraction throughout the engineering program such that students can develop capability to apply mathematics in unseen engineering problems.

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