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MUSIC algorithm for Young Mathematicians and Engineers

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ABSTRACT

CONTEXT

Training of young mathematicians most often focuses only on mathematical and computational methods, without giving students a deeper insight into foundations and context of engineering, which is the most common source of the problems they will be tackling. On the other hand, engineering students are often taught how to manipulate mathematical formulas without having a deeper understanding of the underlying first principles. We believe that there are two reasons for such an unhappy state of modern education. (1) In the modern world, we have become super-specialised, and for one individual to frequently cross the boundaries between fields is not generally perceived as a good career move. Education is often reduced to professional training, where the aim is not to nourish deeper understanding, but just to acquire sufficient skills to perform certain tasks. (2) Specialist textbooks are often replete with discipline specific technical jargon.

PURPOSE OR GOAL

While one might see this situation as the most cost-effective way to provide industry with workforce in the short term, we believe that this approach will eventually erode our ability to make paradigm shifts. Perhaps it is time to take a bit of the attitude of the Renaissance and of Enlightenment and give our education just a broader, more humanistic focus than just providing a skill set.

APPROACH OR METHODOLOGY/METHODS

As an example for how this can be accomplished, we offer our presentation of the MUSIC (Multiple Signal Classification) frequency estimation algorithm, free of any signal processing jargon, not requiring absolutely any knowledge of signal processing but only knowledge of basic linear algebra. We believe that such an approach can give a small hint how to bridge the chasm between education of mathematicians and education of signal processing engineers.

ACTUAL OR ANTICIPATED OUTCOMES

We hope it might help young mathematicians appreciate how spectacularly mathematics is applied in signal processing and we hope that we might surprise young signal processing engineers that one can understand functioning of a key signal processing algorithm relying only on linear algebra.

CONCLUSIONS/RECOMMENDATIONS/SUMMARY

Engineering education should be more integrated, unifying teaching a good deal of science with teaching practical engineering. We must resist the pressures to be “the industry of higher education” and be what we used to be – academia, educating not only practitioners of sophisticated skills but also independent and creative thinkers and innovators.

KEYWORDS

Science in engineering, wholistic education, thinkers, innovators

Introduction

Education for Mathematics and Engineering students often have different focus, thus creating two different worlds for mathematicians and engineers. Students in Engineering often taught to use and manipulate various mathematical formulas with omission of proofs and first principles. Students in mathematics often learn underlying first principles, proofs, mathematical and computational methods with limited exposure in the context of engineering problems. Engineering textbooks, as a rule, try to avoid excursions into mathematical theory as much as possible, while mathematical textbooks usually state the relevant equations without placing them into a proper engineering context. In fact, we feel that engineers and mathematicians mostly live in two separate worlds, often unwilling to cross the boundaries between fields or at least trying to minimise such excursions. Examples are abundant: young mathematicians most often learn harmonic analysis without even a mention of one of the most spectacular applications of harmonic analysis, namely signal processing. On the other hand, young electrical engineers are most often taught the Discrete Fourier Transform (DFT) by simply giving them the relevant formulas, without ever telling them that the DFT amounts to a change of basis of the underlying signal space.

As an approach to reduce this disconnect, we explore the possibility of redesigning learning resources (such as self-paced asynchronous tutorials), which take a commonly used textbook application from Engineering (e.g., a signal processing algorithm from Electrical Engineering) and explain the underlying theory and operation, purely based on mathematical details (such as linear algebra) without any prior knowledge in signal processing. As an example, we present the MUSIC (Multiple Signal Classification) and the root-MUSIC frequency estimation algorithms used in standard signal processing, essentially without any reference to traditional signal processing concepts and without using any signal processing terminology, thus making these algorithms accessible to all students who have studied basic linear algebra, regardless of their field of study. Students of signal processing can also benefit from reading this tutorial which presents very clearly the underlying mathematical foundations of these algorithms.

While the rest of this paper takes the form a mathematical tutorial, the main objective of this paper is to demonstrate how sophisticated technical algorithms in Engineering can be presented in a manner making them accessible to a wide audience, bypassing technical jargon and often requiring minimal background preparation.

Background

In the modern university, students often specialise very early in their studies, and have little substantial engagement with other fields. Henderson and Broadbridge (Henderson & Broadbridge, 2009) describe the state of engineering mathematics education in Australia, including the challenges presented by the diversity of mathematical experience and proficiency in the student body, the need to incorporate computing and statistics, and administrative issues such as class sizes, assessment and budgets. The content and skills taught in these courses are fundamental to the quantitative methods used in various engineering disciplines, as discussed by Maass et al. (Maass, Geiger, Romero Ariza, & Goos, 2019) who write that “STEM education in general, and mathematics education in particular, can contribute to preparing individuals better for twenty-first century challenges”. However, it is well documented that many engineering students struggle in these mathematics courses, and they often perceive the content as unnecessarily abstract and of little relevance to their engineering studies. This issue has received much attention in the literature, such as case studies of the problem-based learning approach by Flegg et al. (Flegg, Mallet, & Lupton, 2012) and Bischof et al. (Bischof, Bratschitsch, Casey, & Rubesa, 2007). In addition, authors such as Ooi (Ooi, 2007) and Klingbeil et al. (Klingbeil, Mercer, Rattan, Raymer, & Reynolds, 2004) write on the matters of what mathematics should be presented to engineering students and at what stage of their studies, with the latter advocating for the traditional calculus sequence to be delayed in favour of teaching freshmen “only the math they really needed to know in order to progress into their sophomore and junior years”.

In this paper, we primarily examine the converse problem: that of demonstrating engineering applications to mathematics students who may be otherwise more interested in mathematics for its own sake. This is much less widely studied, but we believe it to be important in order to both diversify the interests and expertise of mathematics students and foster effective interdisciplinary collaboration. Many mathematics students are not aware of the power of the material they study in solving real-world problems, and here, we demonstrate how they might be introduced to the application of linear algebra in signal processing, without any of the jargon from signal processing, that might be off-putting to a student without any formal training in electrical engineering.

The rest of the paper presents our example tutorial of MUSIC algorithm based on linear algebra.

Example Tutorial on MUSIC Algorithm

The roots of the MUSIC (Multiple Signal Classification) algorithm for frequency estimation of real sinusoids or complex exponentials lie in the early work of Prony (Prony, 1795) and Pisarenko (Pisarenko, 1973) which we present here very briefly and in a very simplified manner.

The methods of Prony and Pisarenko

Let a signal $s(t)$ be a linear combination of n complex exponentials, i.e., of the form

$$s(t) = \sum_{k=1}^{n} A_k e^{j(\omega_k t + \varphi_k)}.$$

(1)

Here, j is the imaginary unit and real numbers A_k , ω_k and φ_k are the amplitude, the frequency, and the phase of the k^{th} component, respectively; we assume that $A_k > 0$ and $-\pi < \omega_k < \pi$, $0 < \varphi_k < 2\pi$ for all $1 \leq k \leq n$.

We also assume that there exists a sequence of samples $s(t+m)$ of such a signal, taken at consecutive instants a unit distance apart, starting with an instant *t*. Let us now form a linear combination of *n + 1* such consecutive samples with coefficients $c_0 , \dots , c_n$ to be specified below. Using (1)**Error! Reference source not found.**, after some simplification we obtain

$$\sum_{m=0}^{n} c_m s(t+m) = \sum_{m=0}^{n} c_m \sum_{k=1}^{n} A_k e^{j(\omega_k (t+m) + \varphi_k)} = \sum_{k=1}^{n} \left(\sum_{m=0}^{n} c_m \left(e^{j\omega_k}\right)^m \right) A_k e^{j(\omega_k t + \varphi_k)}$$

(2)

Consider now a polynomial $P(z)$ with the leading coefficient 1, given by the product $P(z) = \left(z - e^{j\omega_1}\right) \dots \left(z - e^{j\omega_n}\right)$. Let $c_0 , \dots , c_n$ be the coefficients of this polynomial, such that

$$\sum_{m=0}^{n} c_m \, z^m = \prod_{k=1}^{n} \left(z - e^{j\omega_k}\right).$$

Then, since the right-hand side of (2) is of the form $\sum_{k=1}^{n} P\left(e^{j\omega_k}\right) A_k e^{j(\omega_k t + \varphi_k)}$

and since $e^{j\omega_1}, \dots , e^{j\omega_n}$ are the roots of $P(z)$, the right-hand side (2) of will be equal to zero for all real *t*, and vice versa: since complex exponentials with distinct frequencies are linearly independent functions, if the right side of (2) is equal to zero for all real *t*, then $e^{j\omega_k}$ must be the roots of the polynomial $P(z)$. Consequently, in order to find the unknown frequencies ω_k it is enough to find then coefficients $c_0 , \dots , c_n$ such that for all t,

$$\sum_{m=0}^{n} c_m s(t+m) = 0$$

(3)

and then find the roots of the associated algebraic equation $\sum_{m=0}^{n} c_m z^m = 0$; such roots lie on the unit circle and their arguments are the frequencies sought.

To find a non-zero vector $\boldsymbol{c} = c_0 , \dots\ c_n$ such that

(3) holds for all *t*, we instantiate

(3) from $t = 1$ to $= n$, and if $2n$ samples $s(1), s(2), \dots s(2n)$ of the signal $s(t)$ are available, we obtain a system of linear equations in unknown coefficients $c_0 , \dots\ c_n$ of the form

$$\begin{pmatrix} s(1) & s(2) & \cdots & s(n+1) \\ s(2) & s(3) & \cdots & s(n+2) \\ \vdots & \vdots & \ddots & \vdots \\ s(n-1) & s(n) & \cdots & s(2n-1) \\ s(n) & s(n+1) & \cdots & s(2n) \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \\ c_n \end{pmatrix} = \mathbf{0}$$

(4)

which, in absence of any noise, we can solve exactly. This is the Prony method, dating back to year 1795 [**Error! Reference source not found.**], which is summarised in Fig. 1.

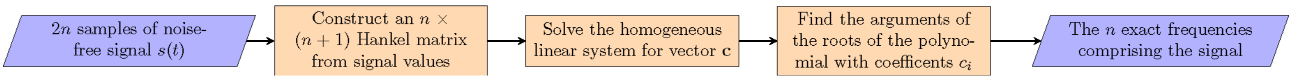

Figure 1: a summary of the Prony method.

In the presence of noise

(3) will not hold exactly; to deal with this problem, we make the system of equations (4) overdetermined. Thus, we will assume that we have $N > 2n$ equidistant samples of the signal $s(t)$. Let us first consider the noise-free case, i.e., assume that $s(t)$ is as in (1) and let us form the following Hankel matrix $\mathbf{M}_p^s$ of size $(N - n) \times (n + 1)$:

$$\mathbf{M}_p^s = \begin{pmatrix} s(1) & s(2) & \dots & s(n+1) \\ s(2) & s(3) & \dots & s(n+2) \\ s(3) & s(4) & \cdots & s(n+3) \\ \vdots & \vdots & \ddots & \vdots \\ s(N-n) & s(N-n+1) & \cdots & s(N) \end{pmatrix}.$$

(5)

If no noise were present,

(3) would imply that the rank of this matrix is equal to n. Consequently, if we consider the singular value decomposition of this matrix, $\mathbf{M}_p^s = \mathbf{U\Sigma V}^*$, the diagonal of $\boldsymbol{\Sigma}$ would consist of $n + 1$ singular values of $\mathbf{M}_p^s$, out of which there would be n non-zero singular values, while the last, which is the smallest of the singular values, would be equal to 0. This is true because singular values on the diagonal of $\boldsymbol{\Sigma}$ are always non-negative reals and are usually ordered in a descending order; thus, the smallest singular value is the rightmost one on the diagonal of $\boldsymbol{\Sigma}$. Matrices $\mathbf{U}$ and $\mathbf{V}$ are both unitary, i.e., their columns represent a set of orthonormal vectors, called the left (the right) singular vectors, respectively. Let $\mathbf{v}_{n+1}$ be the rightmost singular vector which corresponds to the zero singular value of $\mathbf{M}_p^s$. Then, since $\mathbf{V}$ is unitary, $\mathbf{V} * \mathbf{v}_{n+1} = (0,0, \dots\dots ,0,1)^T$ and, since the last entry on the diagonal of Σ is zero, this is easily seen to imply that $\mathbf{M}_p^s \mathbf{v}_{n+1} = \mathbf{U\Sigma V}^* \mathbf{v}_{n+1} = 0$. Thus, the components of the rightmost singular vector $\mathbf{v}_{n+1}$ corresponding to the singular value zero are the required coefficients of a linear combination of the columns of the matrix $\mathbf{M}_p^s$ which is equal to the zero vector.

Assume now that we have $N > 2n$ equidistant samples $f(1), \ldots, f(N)$ of a signal $f(t) = s(t) + v(t)$ which is a sum of a signal $s(t)$ as in (1) and noise $v(t)$. We can still form a Hankel matrix $\mathbf{M}_p^f$ obtained from matrix $\mathbf{M}_p^s$ by replacing samples of $s(t)$ by the corresponding noisy samples of $f(t)$.

Due to the presence of noise, $\mathbf{M}_p^f$ will generally have a full rank of $n + 1$, and thus all singular values of $\mathbf{M}_p^f$ will be non-zero. We now take the rightmost singular vector, which corresponds to the smallest singular value, as an approximation of the rightmost singular vector if no noise were present, and its components $\mathbf{v}_{n+1}(m + 1)$ as approximations of the values of c_m for which equation (3) holds.

As is well known, the right singular vectors of matrix $\mathbf{M}_p^f$ are the eigenvectors of the product matrix $\mathbf{A}_p^f = \left(\mathbf{M}_p^f\right)^* \mathbf{M}_p^f$, where $\mathbf{M}^*$ denotes the conjugate transpose of $\mathbf{M}$. Note that, in our particular case, $(1/N)\mathbf{A}_p^f$ is just the auto-covariance matrix of the samples of the noisy signal $f(t)$. Since matrix $\mathbf{A}_p^f$ is of size $(n + 1) \times (n + 1)$ and since N is usually much larger than n, $\mathbf{A}_p^f$ is of much smaller size than $\mathbf{M}_p^f$. Consequently, finding the eigen decomposition of $\mathbf{A}_p^f$ is a computationally lighter task than finding the singular value decomposition of $\mathbf{M}_p^f$. Note that this benefit, however, is often offset by the cost of the computation of the matrix product $\mathbf{A}_p^f = \left(\mathbf{M}_p^f\right)^* \mathbf{M}_p^f$. In this way we obtain the *Pisarenko frequency estimation algorithm*, which can be summarised as follows and in Fig. 2:

Compute the matrix $\mathbf{A}_p^f = \left(\mathbf{M}_p^f\right)^* \mathbf{M}_p^f$ and obtain its eigenvalue decomposition $\mathbf{A}_p^f = \mathbf{Q\Lambda Q}^*$. Take the rightmost eigenvector $\mathbf{q}_{n+1}$ which corresponds to the smallest eigenvalue of $\mathbf{A}_p^f$ and solve the associated algebraic equation $\sum_{m=0}^{n} \mathbf{q}_{n+1}(m+1)z^m = 0$; the arguments of its n roots are taken as estimates of the unknown frequencies ω_k of the complex exponentials which are the n components of the signal $s(t)$.

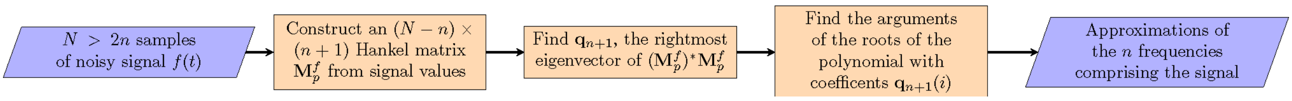

Figure 2: a summary of the Pisarenko method.

MUSIC and root-MUSIC Algorithms

It is well known that the Pisarenko method explained above is not very noise robust and that it often has quite a poor performance. The MUSIC algorithm is a generalization of the Pisarenko method, which significantly improves its noise robustness; it is essentially an averaging procedure of multiple estimates of the frequencies obtained by the Pisarenko method.

To obtain the MUSIC algorithm, let us again first consider the noise-free case and let $\mathbf{M}_m^s$ be the Hankel matrix of shifted consecutive samples with a possibly larger number of columns $K \geq n + 1$, thus of size $(N - K + 1) \times K$,

$$\mathbf{M}_m^s = \begin{pmatrix} s(1) & s(2) & \ldots & s(K) \\ s(2) & s(3) & \ldots & s(K+1) \\ s(3) & s(4) & \vdots & s(K+2) \\ \vdots & \vdots & \ddots & \vdots \\ s(N-K+1) & s(N-K+2) & \cdots & s(N) \end{pmatrix}$$

(6)

and again consider its singular value decomposition, $\mathbf{M}_m^s = \mathbf{U\Sigma V}^*$. In absence of noise equation (3) again implies that the rank of this matrix would be equal to n and thus, only n singular values on the diagonal of $\mathbf{\Sigma}$ would be non-zero. Just as in the case of the Pisarenko method, this would imply that for $K - n$ right singular vectors $\mathbf{v}_i$, $(n < i \leq K)$, which correspond to $K - n$ zero singular values of $\mathbf{M}_m^s$, we would have $\mathbf{M}_m^s \mathbf{v}_i = \mathbf{U\Sigma V}^* \mathbf{v}_i = 0$. Thus, each of the right singular vectors v_i, $n < i \leq K$, produces an equation of the form

$$\sum_{m=0}^{K-1} \mathbf{v}_i(m+1) f(t+m) = 0$$

that holds for samples $f(t), \ldots, f(t+K-1)$ for all integers $1 \leq t \leq N-K+1$. Consequently, in the absence of any noise, for every $n < i \leq K$ and all $1 \leq t \leq N-K+1$, equation (2) would imply

$$\sum_{m=0}^{K-1} \mathbf{v}_i\,(m+1) f(t+m) = \sum_{k=1}^{n} A_k\, e^{j\varphi_k} \left(\sum_{m=0}^{K-1} \mathbf{v}_i\,(m+1)(e^{j\omega_k})^m \right) e^{j\omega_K t} = 0.$$

(7)

If $N - K + 1 \geq n$, since the Vandermonde matrix $\mathbf{V} = \left(\left(e^{j\omega_k}\right)^t : 1 \leq k, t \leq n \right)$ is always nonsingular if all ω_k are distinct, equation (7) would imply that $e^{j\omega_k}$ must be among the $K - 1$ many roots of each of the polynomials $P_i(z)$, where

$$P_i(z) = \sum_{m=0}^{K-1} \mathbf{v}_i(m+1) z^m, \quad n < i \leq K.$$

(8)

Thus, polynomials $P_i(z)$ for all $n < i \leq K$ share the same n roots $e^{j\omega_1}, \ldots, e^{j\omega_n}$ which belong to the unit circle, and each of the polynomials $P_i(z)$ has additional $K - 1 - n$ roots generally not belonging to the unit circle and which are different and specific to each polynomial $P_i(z)$. Thus, if we consider the real valued function

$$F(z) = \sum_{i=n+1}^{K} |P_i(z)|^2 = \sum_{i=n+1}^{K} P_i(z)\overline{P_i(z)},$$

(9)

where $\bar{z}$ denotes the complex conjugation, then this function will have n of its zeros lying on the unit circle, namely $e^{J\omega_k}$ for $1 \leq k \leq n$ and the arguments of these zeros are the unknown frequencies sought.

Clearly, the above no longer holds in the presence of noise. If $f(t) = s(t) + \nu(t)$ where $s(t)$ is of the form given by the right hand side of equation (1) and $\nu(t)$ is noise, then we can form matrix $\mathbf{M}_m^f$ of the same form as matrix $\mathbf{M}_m^s$ but with the samples the noisy signal $f(t)$ in place of the corresponding samples of the noise-free signal $s(t)$. However, such a matrix $\mathbf{M}_m^f$ will generally be of full rank, but we can take its $K-n$ right singular vectors $\mathbf{v}_i$ which correspond to the smallest $K - n$ singular values of $\mathbf{M}_m^f$ as an approximation of such singular vectors, which would correspond to the noise-free matrix $\mathbf{M}_m^s$. We can again form the corresponding polynomials $P_i(z)$ given by equation (8) and function $F(z)$ given by equation (9). Note that $F(z)$ is not a polynomial, due to the presence of the modulus (or the complex conjugation) function. Due to the effects of noise, no roots of polynomials $P_i(z)$ might belong to the unit circle, and also these polynomials might not share the exact same n roots. Thus, function $\Phi(\omega) = F\left(e^{j\omega}\right) = \sum_{i=n+1}^{K} \left|P_i(e^{j\omega})\right|^2$ might not have any real zeros belonging to the interval $[-\pi, \pi]$. Since $\Phi(\omega) > 0$, the MUSIC algorithm thus instead searches for the n values of ω which lie in the interval $[-\pi, \pi]$, where $\Phi(\omega)$ attains n smallest local minima. Fig.4(a) illustrates the equivalent but numerically more convenient search for the n largest local peaks of the reciprocal function

$$R(\omega) = \frac{1}{\sum_{i=n+1}^{K} |P_i(e^{j\omega})|^2}.$$

(10)

To avoid such a numerical search, the root-MUSIC algorithm instead uses the fact that complex numbers z which are close to the unit circle satisfy $\bar{z} \approx z^{-1}$ and thus instead explicitly solves the following equation, conveniently reducible to an algebraic (i.e., polynomial) equation

$$\sum_{i=n+1}^{K} P_i(z)\overline{P}_i(z^{-1}) = 0$$

(11)

where $\overline{P}_i(z)$ denotes the polynomial obtained from the polynomial $P_i(z)$ by taking the complex conjugates of the coefficients of $P_i(z)$.

It is clear that if z_i is a root of equation (11) then so is $\overline{z^{-1}}$; thus, the roots of (11) come in pairs of the form: $\{p_i e^{j\omega}\ ,\ \rho_i^{-1} e^{j\omega}\}$. The root-MUSIC algorithm picks n pairs of such roots that lie closest to the unit circle; the arguments of these pairs are taken as the estimates of the frequencies ω_k $(1 \leq k \leq n)$ of the n components of $s(t)$, as illustrated in Fig.4(b)).

The interested reader can find more details in (Stoica & Moses, 1979) (Pisarenko, 1973).

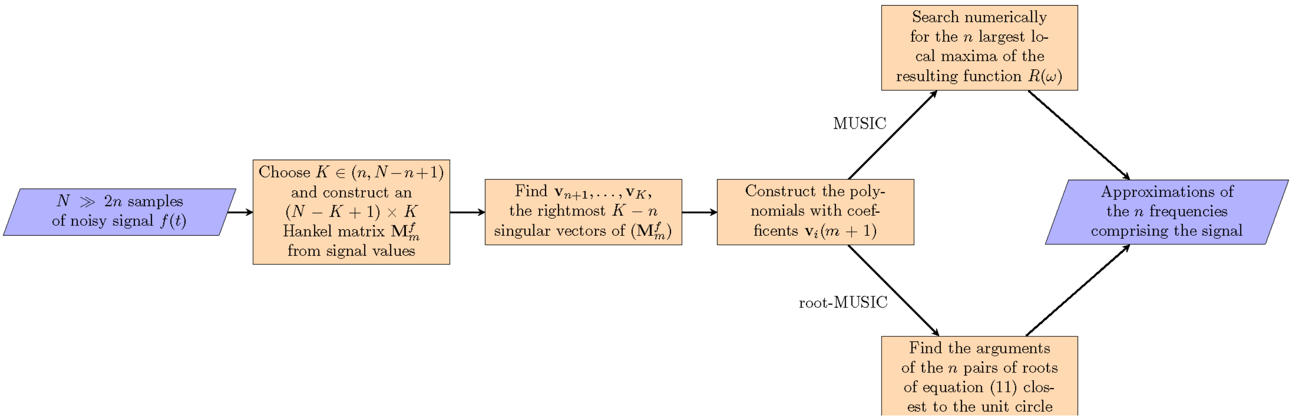

Figure 3: a summary of the MUSIC and root-MUSIC algorithms.

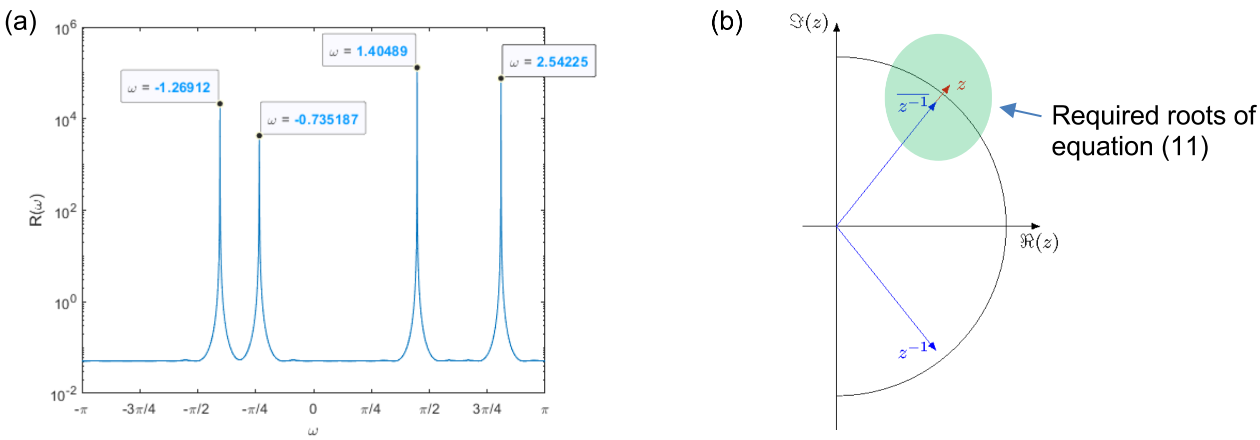

Figure 4: (a) Example peak profile of function $R(\omega)$ given in equation (10); (b) illustration of roots of equation (11) as pairs of complex numbers lying close to unit circle.

Discussion

The tutorial explanations given above were presented to a group of students from which 7 survey responses were recorded. Although the response rate to the feedback survey is quite low, 70% of the responses agreed that “the tutorial provides a clear description about the MUSIC algorithm,” and 56% agreed that “I obtained a good understanding about the MUSIC algorithm after going through the tutorial.” Survey participants also thought that providing more visual interpretation of some of the linear algebra operations will be helpful from a student perspective.

Where traditional methods to teach signal processing algorithms are grounded in terminology from electrical engineering and aim to solve concrete problems, ours is deliberately abstract. We seek to first demonstrate to mathematics students how the singular value decomposition can be used to identify the dominant frequencies in a noisy signal numerically, to be later supplemented by discussion of the application of these methods in digital signal processing. We believe that this approach is different to traditional classroom teaching of such signal processing algorithms.

We envision that this material could be presented in courses on linear algebra, which often introduce the singular value decomposition, but may not contextualise it by demonstrating its application to practical engineering problems. By first relating the theory to a challenging mathematical problem, we aim to inspire mathematics students to consider how their work is related to that undertaken in other disciplines.

This tutorial could also be applied to courses on mathematical computing, as the implementation and analysis of these algorithms in a software package such as MATLAB is an instructive exercise and provides a tangible outcome from linear algebra subroutines.

Conclusion and Future Work

This tutorial presented an approach to reduce the gap between engineering and mathematics education by demonstrating an example design of a mathematics tutorial in the context of electrical engineering. The specific example involved taking an alternative presentation of the conventional MUSIC frequency estimation algorithm used in signal processing, albeit with no signal processing jargon and prior knowledge, but only using the knowledge from linear algebra. Future work includes using employing further visual illustrations of the associated multidimensional vector spaces via interactive simulations and plots, as well as implementing the tutorial in a linear algebra or mathematical computing course.

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